**Parabolic Flow**

Parabolic flows are characterized by a dominant flow direction such that flow quantities are determined by the upstream and cross-stream conditions. No influence is coming from the downstream of the dominant flow direction. There are many parabolic flows in engineering applications such as boundary layer flows, duct flows, jet and mixing layers. Since the predominant flow direction behaves somewhat like time coordinate, 2-dimensional steady parabolic flows can be treated as 1-dimensional transient convection diffusion problem and 3-dimensional steady parabolic flows as 2-dimensional transient convection-diffusion problem.

1. **Two-dimensional parabolic flow**

The mass conservation equation in 2-dimensional steady flow in Cartesian coordinate is

 (1)

The x-momentum equation is

 (2)

For the convection dominated flow, diffusion in the x-direction is small compared to convection, , and the second term in the first bracket can be neglected. Since  y-direction momentum equation is not needed and the pressure gradient along the y-direction is zero. Thus the x-momentum equation reduces to

 (3)

* 1. Finite Volume Representation

Finite volume representation of parabolic flow is obtained by integrating Eqs(1) and (3) over shaded control volume as shown in Fig. 1. All flow properties, density, pressure and x-velocity are defined at the center of control volume surface (circle) and y-velocity is defined at the corner of control volume (diamond). All flow properties are known at x and flow properties at  is to be determined.

Integrating Eq.(1), we get

 (4)

where



Superscript 0 implies known flow properties at x. It is assumed that  evaluated at  prevails over. It is like fully implicit scheme used in time dependent formulation.

s

n





x





y

N

P

S

P0





Fig. 1 Control volume for 2-D parabolic flow in x-direction

Integrating the x-momentum equation over the control volume,, and taking an unit depth in z-direction, we have

 (5)

where



and source term is linearized. The pressure gradient term is excluded from the source term to emphasize its importance.

Now multiplying Eq. (4) by  and subtract it from Eq. (5) and introducing



the following finite volume representation of x-momentum equation results

 (6)

where

 (6a)

 (6b)

 (6c)

 (6d)

 (6e)

Diffusion conductance and convection strength are evaluated as before and power law scheme is used for the interface flux approximation. For example, at the north interface,

 (6.f)

We observe a strong similarity between the 2- D parabolic flow and 1-D transient convection-diffusion problem in y-direction. Table 1 shows differences between two formulations.

Table 1

|  |  |  |
| --- | --- | --- |
|  | 2-D Parabolic | Transient 1-D |
|  |  |  |
|  |  |  |
|  |  |  |

Once velocity u is found by solving Eq.(6) by TDMA, y-direction velocity is calculated by using the continuity, i.e., Eq. (4).

 (7)

Eq.(7) provides v at all y starting with the known boundary value of v at y=0.

Unlike time dependent convection-diffusion formulation, there is no need to iterate within a step as long as it is not too large. The pressure gradient in the x-direction must be prescribed in the source term.

**1.2 External Boundary Layer**

In the external boundary layer, pressure gradient along the x-direction can be determined by the free stream velocity.

 (8)

where (x) is the free stream velocity.



**1.2.1 Momentum boundary layer over a flat plate**

As an example of external boundary layer, consider air flow over a flat plate as shown in the figure. Air properties at 300 K are used. Since flat plate, the pressure gradient is zero. In selecting calculation domain, x distance is limited by laminar flow constraint and maximum y distance should be well above the momentum boundary layer.

x



Calculation domain

Fig. 2 Momentum boundary layer over a flat plate

Y-direction grid is clustered near the solid wall to capture boundary layer thickness and smaller step size is used near x=0 to capture rapid change in flow quantities.

Numerical results in terms of local shear stress is calculated and compared with known Blasius solution (table 2). Also non-dimensional velocity  is compared with known solutions (Fig. 3). Numerical local shear stress agrees well with the Blasius solution except near the leading edge. This is expected since boundary layer solution is not valid near the leading edge. A good agreement is shown for the velocity profile.

Table 2 Local shear stress comparison

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| x(m) | 0.231 | 0.451 | 0.671 | 1.111 | 1.21 |
| Blasius | 0.0429 | 0.0307 | 0.0252 | 0.0196 | 0.0187 |
| Present | 0.0405 | 0.0299 | 0.0250 | 0.0197 | 0.0189 |

Fig. 3 

**1.2.2 Thermal boundary layer**

Using enthalpy as the dependent variable, the steady energy equation in x-direction is



In the boundary layer, . For slow speed flow viscous dissipation  Thus above equation reduces to

 (9)

Eq.(9) is parabolic in x-direction and can be solved by parabolic flow approximation. The first term in RHS of Eq. (9) is generally very small compared with enthalpy flux and can be also neglected in slow speed flow. This assumption is consistent with neglecting viscous dissipation term in Eq.(9).

By solving momentum boundary layer and thermal boundary layer simultaneously, combined developing boundary layer flow can be analyzed. As an example of combined boundary layer flow, consider previous problem with the solid wall temperature is raised to 700 K. Numerical results are compared with exact solution in terms of local Nusselt number.

Table 3 Local Nusselt number comparison

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| x(m) | 0.231 | 0.451 | 0.671 | 1.111 | 1.21 |
| Blasius | 71.56 | 99.97 | 121.97 | 156.94 | 163.79 |
| Present | 67.16 | 96.45 | 119.03 | 154.80 | 161.78 |

The agreement becomes better at locations away from the leading edge.

A more general external boundary layer problem with non-zero pressure gradient can be easily analyzed by modifying the pressure term using free stream velocity, . The control of momentum and thermal boundary layers can be achieved by blowing and suction at the plate. This can be accomplished by specifying non-zero v at the plate surface. Some of these applications are left as exercise problems.

* 1. **Internal Flow**

Another example of 2-dimensional parabolic flow is steady flow through ducts. In this case, pressure gradient can not be determined by the free stream velocity. Instead mass conservation is used to find the pressure gradient. Since exact pressure gradient is not known, calculation begins with an assumed pressure gradient at x. That is



The x-momentum equation with the assumed pressure gradient is

 (10)

where \* represent velocity obtained by using the assumed pressure gradient. In Eq.(10) source term is modified from that in Eq.(6) by taking out the pressure gradient term. By solving Eq.(10 ), we obtain .

The exact velocity and pressure gradient are

**** (11)

where prime denotes velocity deviation and pressure gradient deviation.

Substituting Eq.(11) into Eq.(6) and using Eq.(10), we get



Now we assume that

 (12)

Eq.(12) relates pressure correction to the velocity correction.

The mass conservation requires that



Substitute



into the last equation and solve for the pressure gradient correction term, noting that it is independent of y, we have

 (13)

Substituting Eq.(13) into Eq.(12),

 (14)

The first bracket term in RHS of Eq.(14) is dependent on y while the second bracket term is not.

By adding velocity and pressure corrections given by Eqs.(13) and (14) respectively, we get corrected velocity and pressure at . The y-velocity is obtained as in the external parabolic flow.

1.3.1 Entrance flow between two parallel plates

Consider air entering parallel plates as shown in Fig. 4. Air velocity is 0.5 m/s and its temperature is 300 K while the plate temperature is at 700 K. The energy equation, Eq.(9) is also applicable to this situation. Appropriate modifications were added to the program used for the external flow to account for the pressure change. The local skin friction coefficient and local Nusselt number are compared with the known analytic solutions.

Te=300 K





x=3 m





D=0.05 m

Fig. 4 Developing flow between two parallel plates

x

The local skin friction coefficient is defined

 (15)

Sparrow [[1]](#footnote-1) showed by using approximate integral method that

 (16)

where  and  is the centerline velocity. Numerical skin friction is estimated by using Eq.(15) where wall shear stress is obtained from numerical velocity profile. The centerline velocity is expressed as an implicit function of distance x in analytical solution. Instead, in the present calculation, numerically obtained centerline velocity is used in Eq.(16). Fig. 5 shows a good agreement in the local skin friction coefficient. Fig. 6 shows local Nusselt number comparison for thermally developing flow with constant wall temperature. Agreement with a reference[[2]](#footnote-2) is not as good in the entrance region as the fully developed region. This is partially due to the approximation involved in analytical solution using boundary layer approximation.

Fig. 5 Local Skin friction between two parallel plates

Fig. 6 Local Nusselt number between parallel plates (constant wall temperature)

**1.3.2. Entrance flow in a circular duct**

It can be shown that mass, momentum and energy conservation equations parabolic flow through a circular ducts are

 (17a,b,c)



In Eq.(17) y is the radial coordinate and x is the axis. Integrating Eq.(17 ) over a control volume,, and taking unit radian in the angular direction, we obtain a finite volume representation of Eq.(17) similar to Cartesian case except the control volume size and its surface area are dependent on the radial coordinate as seen in previous chapters.

Consider air entering a circular pipe at constant velocity of 0.2 m/s. The air temperature is 300 K and the duct wall is constant at 700 K. Numerical results are compared with known data in terms of local skin friction coefficient and local Nusselt number. This example is similar to the previous case except the geometry.

P

Fig. 7 Local skin friction coefficient and pressure variation in the entrance region of a circular tube

Fig. 8 Local Nusselt number in the entrance region of a circular tube (constant wall temperature)

Numerical results are in good agreement with reference values in both skin friction and Nusselt number. The pressure variation is nonlinear in the developing region (x/D/Red= ~0.01) and becomes linear in the hydro-dynamically fully-developed region.

1. Three-Dimensional Parabolic Flow

Three–dimensional parabolic flows are also characterized by a dominant flow direction. No reverse flow is allowed and flow condition is influenced by the upstream and cross stream conditions. Examples of 3-dimensional parabolic flow are 3-dimensional boundary layer flows, flow through non-circular ducts and flow through duct with curvatures.

To extend the numerical formulation of 2-D parabolic flow to 3-D parabolic flow, consider a 3-D steady flow through a rectangular duct.

y

z

x

x



y

z

Fig. 8 Three dimensional parabolic flow in a rectangular duct showing dominant flow in x-direction and secondary flows on y-z plane

The conservation of mass, momentum in x, y, and z-directions are

 (18)

 (19)

 (20)

 (21)

There are two pressure fields in these equations. The pressure gradient in the x-momentum equation is the averaged pressure over the cross section at each axial direction and is independent of y and z-coordinates. The pressure gradients in the y and z-direction are responsible for the cross flow velocity fields. This decoupling of pressure gradient into two parts: a dominant flow direction and a cross flow direction is an important concept in 3-dimensional parabolic flow formulation.

Integrating Eq.(19) over a control volume (see Fig. 9) , , we obtain finite volume representation of x-momentum equation. Using a guessed dominant pressure gradient, we get,  at the down stream location as we did in 2-D parabolic procedure. Similarly, mass conservation requirement provides the pressure correction term as

 (22)

Then the velocity correction is given by

 (23)

x

y

z

E

W

N

S

∆x

P

Fig. 9 A control volume module for u-velocity

The cross flow velocities, v and w, are defined at the nominal control volume surface defined for u, p and density and staggered control volumes are used for the integration of cross flow momentum equation. Fig. 10 shows staggered grid arrangement for cross flow velocities.

∆y

∆z

z

y

Fig. 10 Staggered grids for the cross flow velocities on the y-z plane: v (////); w(\\\\)

Integrating y-momentum equation over a staggered control volume (shown with //// in Fig. 10), we get a finite volume representation as

 (24)

where a superscript v denotes quantities associated with y-momentum equation. All coefficients in Eq.(24) have the usual meanings as discussed before. For example,

 (25)

Similarly z-momentum equation is integrated over a control volume (shown with \\\\ in Fig. 10), we have

 (26)

Eqs.(24) and (25) can not be solved until the cross flow pressure fields are known. To start the computation, we start with an assumed cross pressure field p\* everywhere. With a guessed pressure fields, Eq.(24) is

 (27)

Subtracting Eq.(27) from Eq.(24), we have

 (28)

where the primed quantities are deviations from the correct values. Eq.(28) relates the pressure correction to the y-velocity correction. Similarly for the z-velocity correction

 (29)

The correct velocity and pressure fields are then

 (30)

To obtain the pressure correction, we now turn to the requirement of mass conservation.

Ultimately the correct velocity fields must satisfy the mass conservation, Eq.(18). Rewriting Eq.(18),

 (31)

The RHS of Eq.(31) is a known quantity from the x-momentum equation and Eq. (31) is a 2-dimensional continuity with a known source term. Integrating Eq.(31) over a nominal control volume, we have

 (32)

Substituting Eq.(30) into Eq.(32) and using Eqs.(28) and (29), we get the pressure correction equation,

 (32)

where

 (33a)

 (33b)

 (33c)

 (33d)

 (33e)

and

 (33f)

Eq.(33f) is the residual mass in a nominal control volume due to incorrect cross flow velocities. Eq.(32) is solved repeatedly until the mass residual in each control volume becomes small.

In summary, the procedure to obtain the cross flow velocities is:

1. Guess cross flow pressure fields at x+∆x. The best guess could be the value at the immediate upstream value at x.
2. Solve w\* and v\* using z-momentum and y-momentum equations with guessed pressure.
3. Evaluate the mass residual using Eq.(33f). If it is less than an error tolerance, no iteration is needed. If not, solve the pressure correction equation, Eq. (32) and update the pressure and the cross flow velocities using Eqs. (28)-(30) and go to step (2).
4. Repeat this process until mass residual in all control volume reduces below an error tolerance.

**2.1 Entrance flow in a rectangular duct**

Consider a micro-channel flow through a duct with constant rectangular cross sectional area as shown in Fig. 11. The width (in z-direction) and the height (in y-direction) are 0.48 mm and 0.46 mm, respectively. The water entering the channel has uniform velocity of 1.1 m/s. All fluid properties are evaluated at 300 K and remain constants. No heat transfer is considered. The flow is laminar with Red=603. A numerical program based on 3-D parabolic flow as discussed in previous section is used. Fig. 12 shows the numerical results in terms of the mean pressure and the local friction factor distribution along the channel. Results agree well with known data[[3]](#footnote-3) at the fully developed region.

z

x

y

Fig. 11 Developing flow through a rectangular micro-channel

f Red

Fig. 12 Pressure and friction factor in a developing rectangular channel flow

Fig. 13 shows the axial velocity distribution at x=0.025 m. The maximum velocity is at the center and is about 2.06 m/s.

3dpara1

Fig. 13 Axial velocity distribution at x=0.025 m. (maximum at the center)

Fig. 14 shows the cross flow pressure distribution and the cross flow velocity vectors at x=0.025 m. The hydrodynamic entrance length for a rectangular duct2 is. Thus there should be no cross flow velocity components at x=0.025 m. However, the magnitude of cross flow velocity is very small compared with the axial velocity. The maximum cross flow speed is about 0.054 m/s which is negligible compared to the axial velocity. The magnitude of the cross flow pressure field (p) is in the order of 0.002 Pa. This is a small pressure compared with the averaged pressure along the axis () which is in the order of 3000 Pa.

3dpara2

Fig. 14 Cross flow pressure distribution and cross flow velocity vectors at x=0.025 m

**Problems**

1. Consider a developing laminar flow between two concentric cylinders as shown in the figure. The flow enters the concentric tube annulus at a uniform velocity. The inside cylinder is insulated and the outside cylinder is at a constant temperature which is higher than the temperature of the fluid.

Determine the developing combined momentum and thermal entrance flow using 2dparay.for. You may choose any fluid and any dimensions for the cylinders as long as the flow remains laminar and Di/Do=0.10.

z

r

Di

Do

1. Plot local skin friction times the Reynolds number as a function of axis. Use hydraulic diameter for the Reynolds number.
2. Plot the pressure variation along the axis.
3. Plot local Nusselt number, Nux, along the axis.
4. In the hydrodynamically-fully-developed region, compare your velocity profile with the exact solution (Eq. (6.155) in Munson, Young and Okiishi,4th Ed.).
5. In the thermally-fully-developed region, compare your local Nusselt number with a known data (Table 8-2, Incropera and DeWitt).

2. Consider developing momentum boundary layer flow over a flat plate with weak blowing with a vertical velocities ranging from 0.001 - 0.005 m/s. Calculate the local skin friction  and velocity profile  Compare with the known solutions.

3. Free stream velocity of laminar flow over a wedge is given by



where the exponent m is related to the wedge angle  by



and C is a constant.



x

y



Plot local wall shear stress as a function of x and plot non-dimensional velocity profile as a function of similarity variable for several values of wedge angle, β. Compare with the available data. ( Convective Heat Transfer, Burmeister, pp. 297-307)

1. E. M. Sparrow, “Analysis of laminar forced convection heat transfer in the entrance region of flat rectangular ducts”,NACA TN3331,1955. [↑](#footnote-ref-1)
2. R. K. Shah and A. L. London, Laminar forced convection in ducts, Supplement 1 to Advances in Heat Transfer, Academic Press, New York, 1978. [↑](#footnote-ref-2)
3. D. Liu and S. Garimella, “Investigation of Liquid Flow in Microchannels”, AIAA20002-2776, 8th AIAA/ASME Joint Thermophysics and Heat Transfer Conference, June 2002, St. Louis, Missouri. [↑](#footnote-ref-3)